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ORIGINAL RESEARCH

Two effective total ranking models for preference voting and aggregation

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Abstract Data envelopment analysis (DEA), a useful assessment tool, has been used to solve the problem of preference voting and aggregation which requires the determination of the weights associated with different ranking places. Instead of applying the same externally imposed weighting scheme to all candidates, DEA models allow each candidate to choose his/her own weights to maximize his/her own overall ratings subject to certain conditions. This paper proposes two new models to assess the weights. The proposed models are linear programming, which determine a common set of weights for all the candidates. The proposed models are examined with two numerical examples and it is shown that the proposed models can not only choose a winner, but also give a full ranking of all the candidates.

Keywords Scoring rules · Data envelopment analysis · Preference voting

Introduction

In a preferential voting system, each voter selects a subset of the candidates and places them in a ranked order. The key issue of the preference aggregation in a preferential voting system is how to determine the weights associated with different ranking places. To avoid the subjectivity in determining the weights, data envelopment analysis (DEA) is used in Cook and Kress [2] to determine the most favorable weights for each candidate. Different candidates utilize different sets of weights to calculate their total scores, which are referred to as the best relative total scores and are all restricted to be ≤ 1. The candidate with the biggest relative total score of one is said to be DEA efficient and may be considered as a winner. This approach proves to be effective, but very often leads to more than one candidate to be DEA efficient. To choose a winner from among the DEA-efficient candidates, Cook and Kress [2] suggest maximizing the gap between the weights so that only one candidate is left as DEA efficient. Green et al. [3] suggest using the cross-efficiency evaluation method in DEA to choose the winner. Noguchi et al. [7] also utilize cross-efficiency evaluation technique to select the winner, but present a strong ordering constraint on the weights. Hashimoto [6] proposes the use of the DEA exclusion model (i.e. super-efficiency model) to identify the winner. Obata and Ishii [8] suggest excluding non-DEA-efficient candidates and using normalized weights to discriminate the DEA-efficient candidates. Their method is subsequently extended to rank non-DEA-efficient candidates by Foroughi and Tamiz [4] and Foroughi et al. [5]. Recently, Wang et al. [10] also propose three new models to assess the weights and rank the candidates. Two of them are linear programming models which determine a common set of weights for all the candidates considered and the other is a nonlinear programming model that determines the most favourable weights for each candidate. But, Wang et al. have not taken care of about making the weight of a certain rank zero means that we throw away the corresponding part of the obtained data. In actual applications, making the weight of a certain rank zero means that we throw away the corresponding part of the obtained rank voting data. Their incorrect model also used in [9]. To avoid possible more misapplications or spread in the future, we present in this paper two improved DEA models to determine the weights

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of ranking places. The proposed models are simple, and each of them can lead to a stable full ranking for all the candidates considered. This will be illustrated with two numerical examples.

The rest of the paper is organized as follows. In the next section, we develop the models for preference aggregation to assess the weights associated with different ranking places. We then examine two numerical examples using the proposed models to illustrate their applications and show their capabilities of identifying the winner and producing a stable full ranking for all the candidates considered. Finally, we conclude the paper.

The models

Let w_j be the relative importance weight attached to the jth ranking place ($j = 1, \ldots, m$) and v_{ij} be the vote of candidate i being ranked in the jth place. The total score of each candidate is defined as

$$Z_i = \sum_{j=1}^{m} v_{ij} w_j, \quad i = 1, 2, \ldots, n \tag{1}$$

which is a linear function of the relative importance weights. Once the weights are given or determined, candidates can be ranked in terms of their total scores.

To determine the score of each candidate, Wang et al. [10] suggest the following DEA model, which maximizes the minimum of the total scores of the n candidates and determines a common set of weights for all the candidates:

$$\begin{aligned} &\max \quad \alpha \\ &\text{s.t.}\, \alpha \le Z_i = \sum_{j=1}^{m} v_{ij} w_j \le 1, \quad i = 1, \ldots, n \\ &w_1 \ge 2w_2 \ge \cdots \ge m w_m \ge 0, \end{aligned} \tag{2}$$

where $w_1 \ge 2w_2 \ge \cdots \ge m w_m \ge 0$ is the strong ordering constraint on decision variables.

But, Wang et al. have not taken care of about making the weight of a certain rank zero. In actual applications, making the weight of a certain rank zero means that we throw away the corresponding part of the obtained rank voting data. Here, using an example we show this assertion.

Example 1 Consider the example in which 20 voters are asked to rank two out of four candidates A–D on a ballot. The votes each candidate receives are shown in Table 1.

If the model (2) is employed to solve the example then we get $\alpha^* = 0.4286, w_1^* = 0.1429$ and $w_2 = 0.0000$. As we see the weight of second place is zero, which means that the second place vote does not have any meaning. In actual applications, making the weight of a certain place vote zero means that we throw away the corresponding part of the obtained data.

Table 1 Votes received by four candidates

Candidate	First place	Second place
A	7	7
B	7	8
C	3	3
D	3	2

In what follows, we present our models, which avoid producing a zero weight for a certain place vote and make full use of all the data.

First model

Consider the following model,

$$\begin{aligned} &\max \quad \alpha \\ \text{s.t.} \quad &\alpha \le Z_i = \sum_{j=1}^{m} v_{ij} w_j \le 1, \quad i = 1, \ldots, n \\ &w_1 \ge 2w_2 \ge \cdots \ge m w_m, \\ &w_m \ge \varepsilon \end{aligned} \tag{3}$$

As a theoretical construct, ε provides a lower bound for scoring of grades to keep them away from zero. Hence, the following LP is proposed to determine the ε.

$$\begin{aligned} &\varepsilon^* = \max \quad \varepsilon \\ \text{s.t.} \quad &\sum_{j=1}^{m} v_{ij} w_j \le 1, \quad i = 1, \ldots, n \\ &w_1 \ge 2w_2 \ge \cdots \ge m w_m, \\ &w_m - \varepsilon \ge 0 \end{aligned} \tag{4}$$

It is clear that $\varepsilon = 0, \forall j : w_j = 0$ is a feasible solution to the model (4).

Lemma 1 The optimal value of model (4) is >0, that is $\varepsilon^* > 0$.

Proof The dual of model (4) is as follows:

$$\begin{aligned} &\min \quad \sum_{i=1}^{n} \theta_i \\ \text{s.t.} \quad &\sum_{i=1}^{n} v_{i1} \theta_i - \delta_1 = 0 \\ &\sum_{i=1}^{n} v_{ij} \theta_i + j\delta_{j-1} - j\delta_j = 0, \quad j = 2, \ldots, m-1 \\ &\sum_{i=1}^{n} v_{im} \theta_i + m\delta_{m-1} - \delta_m = 0 \\ &\delta_m = 1 \\ &\theta_i, \delta_j \ge 0, i = 1, \ldots, n; \quad j = 1, \ldots, m \end{aligned} \tag{5}$$

By contradiction assume that $\varepsilon^* = 0$. Hence, $\theta^* = \sum_{i=1}^{n} \theta_i^* = 0$. Therefore according to the constraints of model (5), for all $j = 1, \ldots, m$, we have $\delta_j = 0$ which contradicts to the last constraint of model (5), So $\varepsilon^* = \theta^* > 0$. □

Lemma 2 $\varepsilon^* \leq \min_{1 \leq i \leq n} \left\{ \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}} \right\}$.

Proof From $w_1 \geq 2w_2 \geq \cdots \geq mw_m$, we have $w_j \geq \frac{mw_m}{j}$. Thus

$$Z_i = \sum_{j=1}^{m} v_{ij} w_j \geq \sum_{j=1}^{m} \frac{mw_m}{j} v_{ij} = mw_m \sum_{j=1}^{m} \frac{v_{ij}}{j}$$

But according to the constraints of model (3) for each $i = 1, \ldots, n$ we have $Zi \leq 1$. Therefore $mw_m \sum_{j=1}^{m} \frac{v_{ij}}{j} \leq 1$, or $w_m \leq \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}}, i = 1, \ldots, n$. On the other hand according to the last constraint of model (3), $w_m \geq \varepsilon$. Hence $\varepsilon \leq \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}}, i = 1, \ldots, n$. So, $\varepsilon^* \leq \min_{1 \leq i \leq n} \left\{ \frac{1}{m \sum_{j=1}^{m} \frac{v_{ij}}{j}} \right\}$. □

Theorem 1 *The optimal value of model (4) is >0 and bounded.*

Proof The proof is clear using the above lemmas.

Theorem 2 *The model (4) and the following model are equivalent:*

$$\begin{aligned} & \max \quad \mathrm{w_m} \\ \text{s.t.} \quad & \sum_{j=1}^{m} v_{ij} w_j \leq 1, \quad i = 1, \ldots, n \\ & w_1 \geq 2w_2 \geq \cdots \geq mw_m \geq 0 \end{aligned} \tag{6}$$

Proof From the last constraint of model (5) we have $\delta_m = 1$. Hence, we can write the model (5) as follows:

$$\begin{aligned} & \min \quad \sum_{\mathrm{i}=1}^{\mathrm{n}} \theta_{\mathrm{i}} \\ \text{s.t.} \quad & \sum_{i=1}^{n} v_{i1} \theta_i - \delta_1 = 0 \\ & \sum_{i=1}^{n} v_{ij} \theta_i + j\delta_{j-1} - j\delta_j = 0, \quad j = 2, \ldots, m-1 \\ & \sum_{i=1}^{n} v_{im} \theta_i + m\delta_{m-1} = 1 \\ & \theta_i, \delta_j \geq 0, i = 1, \ldots, n; \quad j = 1, \ldots, m-1 \end{aligned} \tag{7}$$

Now, the dual of model (7) is as follows:

$$\begin{aligned} & \max \quad \mathrm{w_m} \\ \text{s.t.} \quad & \sum_{j=1}^{m} v_{ij} w_j \leq 1, \quad i = 1, \ldots, n \\ & w_1 \geq 2w_2 \geq \cdots \geq mw_m \geq 0 \end{aligned} \tag{8}$$

But we know that the dual of the dual is primal, thus the above model is the same as model (4). □

By solving model (4) for data of Table 1, we have $\varepsilon^* = 0.04545$. If this ε^* is employed to solve the model (3), for example, 1 we get $\alpha^* = 0.36364, w_1^* = 0.9091$ and $w_2^* = 0.04545$. Hence, the ranking of the four candidates is as: $B \succ A \succ C \succ D$. Therefore, candidate B is the winner.

Second model

To avoid producing a zero weight for the last ranking place and make full use of all the data, we propose that model (2) be modified as

$$\begin{aligned} & \max \quad \alpha + \mathrm{w_m} \\ \text{s.t.} \quad & \alpha \leq Z_i = \sum_{j=1}^{m} v_{ij} w_j, \quad i = 1, \ldots, n \\ & w_1 \geq 2w_2 \geq \cdots \geq mw_m \geq 0, \\ & \sum_{j=1}^{m} w_j = 1 \end{aligned} \tag{9}$$

which maximizes α and the minimum weight w_m at the same time. By solving the above modified model for example 1, we got the unique optimal solution $\alpha^* = 1.0000, w_1^* = 0.6667$ and $w_2^* = 0.3333$. The ranking of the four candidates generated by the above model is $B \succ A \succ C \succ D$, which consistent with model (3).

Numerical examples

In this section, we examine two numerical examples using the proposed models to illustrate their applications and show their capabilities of choosing the winner and ranking candidates.

Example 2 Consider the example investigated by Cook and Kress [2] and Wang et al. [10] in which 20 voters are asked to rank four out of six candidates A–F on a ballot. The votes that each candidate receives are shown in Table 2.

By solving (4), we get $\varepsilon^* = 0.02727$. Now the model (3) yields $\alpha^* = 0.43636, w_1^* = 0.10909, w_2^* = 0.05455, w_3^* = 0.03636$ and $w_4^* = 0.02727$. Solving the model (8), we have: $\alpha^* = 1.0000, w_1^* = 0.4800, w_2^* = 0.2400, w_3^* = 0.1600$ and $w_4^* = 0.1200$. The rankings of the six candidates produced

Table 2 Votes received by six candidates

Candidate	First place	Second place	Third place	Fourth place
A	3	3	4	3
B	4	5	5	2
C	6	2	3	2
D	6	2	2	6
E	0	4	3	4
F	1	4	3	3

Table 3 Scores and rankings of the six candidates by proposed models

Candidate	First model		Second model	
	Score	Rank	Score	Rank
A	0.71818	4	3.16000	4
B	0.94546	2	4.16000	2
C	0.92727	3	4.08000	3
D	1.00000	1	4.40000	1
E	0.43636	6	1.92000	6
F	0.51818	5	2.28000	5

Table 4 Votes received by seven candidates

Candidate	First place	Second place
A	32	10
B	28	20
C	13	36
D	20	27
E	27	19
F	30	8
G	0	30

Table 5 Scores and rankings of the seven candidates by proposed models

Candidate	First model		Second model	
	Score	Rank	Score	Rank
A	0.97369	2	24.66677	2
B	1.00000	1	25.33333	1
C	0.81579	6	20.66667	6
D	0.88158	5	22.33333	5
E	0.96053	3	24.33333	3
F	0.89474	4	22.66677	4
G	0.39474	7	10.00000	7

by the two models are shown in Table 3, from which it is clear that the two models all lead to the same ranking, that is, $D \succ B \succ C \succ A \succ F \succ E$. Therefore, candidate D is the winner.

Example 3 Consider the example investigated by Obata and Ishii [8] and Foroughi and Tamiz [4], in which seven candidates A–G are ranked. Table 4 shows the votes each candidate receives in the first two places.

Using the model (4), we have $\varepsilon^* = 0.01316$. Now, by solving model (3) we get $\alpha^* = 0.39474$, $w_1^* = 0.02632$ and $w_2^* = 0.01316$. Solving the model (9), we have: $\alpha^* = 1.0000$, $w_1^* = 0.6667$ and $w_2^* = 0.3333$. The rankings of the seven candidates generated by the two models are shown in Table 5. As can be seen from Table 5, our models lead to the same ranking, $B \succ A \succ E \succ F \succ D \succ C \succ G$. Therefore, candidate B is the winner.

Conclusion

We discussed the applicability of the ranking method proposed by Wang et al. By using DEA, we determine the weights from rank voting data. Their model, gives rise to the case such that the data of some rank are ignored. Thus, we analyze the procedure to determine weights, and proposed two extended models for preference voting and aggregation. The contribution of this paper is to maintain the effects of all data in the final solution, an improvement over the model proposed by Wang et al.

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